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# Tableau models for Schubert polynomials

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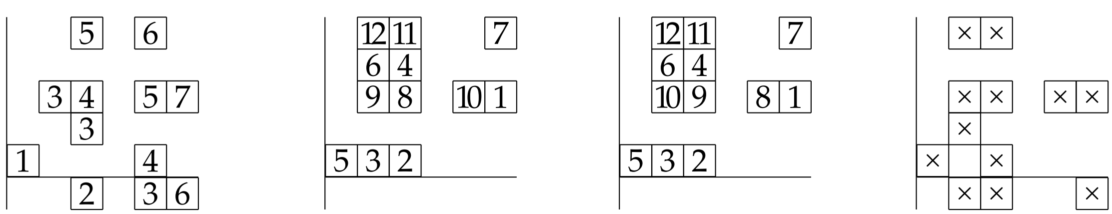
**Abstract.** We introduce several new (and recall one old) tableau models for Schubert polynomials. Applications include a bijective proof of Kohnert's rule.

**Résumé.** Nous introduisons plusieurs nouveaux (et rappelons un ancien) modèles de tableau pour les polynômes de Schubert. Les applications comprennent une preuve bijective de la règle de Kohnert.

**Keywords:** Rothe tableaux, balanced tableaux, Schubert polynomials

## 1 Introduction

Lascoux and Schützenberger [6] defined polynomial representatives for the Schubert classes in the cohomology ring of the complete flag variety that have beautiful algebraic and combinatorial properties. The structure constants for these *Schubert polynomials* give intersection numbers for Schubert varieties, so there is much to be gained from developing models that might facilitate computations or, better, combinatorial formulas for these numbers. In this abstract, we survey new and old combinatorial models for Schubert polynomials, and we relate them to one another with simple bijections between the underlying combinatorial sets. These models are illustrated in Figure 1.



§ 3 reduced diagram  § 4 balanced tableau  § 5 Rothe tableau  § 6 Kohnert diagram

**Figure 1:** Combinatorial models for the reduced expression $(5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)$.

In Section 2 we review of the monomial expansion of Schubert polynomials due to Billey, Jockusch, and Stanley [2]. We reformulate this using the fundamental slide

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polynomials of Assaf and Searles [1] to realize Schubert polynomials as the generating polynomials for reduced expressions. We then present our models.

Section 3 introduces the new model of reduced diagrams, which resembles the pipe dream model, though with a more direct connection to reduced expressions. Section 4 recalls the balanced tableaux model of Edelman and Greene [3] and gives a natural bijection with reduced diagrams by pushing boxes of the latter up. Section 5 introduces the new model of Rothe tableaux, which resemble balanced tableaux but stem from not allowing boxes of a reduced diagram to move right when pushed down, as indicated in the bijection with balanced tableaux. This leads naturally to Section 6, where we recall Kohnert diagram [5] and give a simple bijection with Rothe tableaux.

## 2 Schubert polynomials

Lascoux and Schützenberger [6] originally defined polynomial Schubert polynomials via divided difference operators, with a combinatorial model given by Billey, Jockusch, and Stanley [2]. We begin our treatment with the latter formulation, and we refer the reader to Macdonald [7] for a beautiful and thorough treatment of the underlying combinatorics.

A *reduced expression* is a sequence $\rho = (i_k, \ldots, i_1)$ such that the permutation $s_{i_k} \cdots s_{i_1}$ has $k$ inversions, where $s_i$ is the simple transposition that interchanges $i$ and $i+1$. Let $R(w)$ denote the set of reduced expressions for $w$. For example, the elements of $R(153264)$ are shown in Figure 2.

$$\begin{array}{cccccc} (5,3,2,3,4) & (5,2,3,2,4) & (5,2,3,4,2) & (3,5,2,3,4) & (3,2,5,3,4) & (3,2,3,5,4) \\ (2,5,3,4,2) & (2,3,5,4,2) & (2,5,3,2,4) & (2,3,5,2,4) & (2,3,2,5,4) & \end{array}$$

**Figure 2:** The set of reduced expressions for 153264.

For $\rho \in R(w)$, say that a strong composition $\alpha$ is *$\rho$-compatible* if $\alpha$ is weakly increasing with $\alpha_j < \alpha_{j+1}$ whenever $\rho_j < \rho_{j+1}$ and $\alpha_j \leq \rho_j$.

**Definition 2.1** ([2])**.** The Schubert polynomial $\mathfrak{S}_w$ is given by

$$\mathfrak{S}_w = \sum_{\substack{\rho \in R(w) \\ \alpha\ \rho-\text{compatible}}} x_{\alpha_1} \cdots x_{\alpha_{\ell(w)}}, \tag{2.1}$$

where the sum is over compatible sequences $\alpha$ for reduced expressions $\rho$.

We harness the power of the *fundamental slide polynomials* of Assaf and Searles [1] to re-express Schubert polynomials as the generating function for reduced expressions. Given a weak composition $a$, let $\mathrm{flat}(a)$ denote the strong composition obtained by removing all zero parts.

**Definition 2.2** ([1]). For a weak composition $a$ of length $n$, define the *fundamental slide polynomial* $\mathfrak{F}_a = \mathfrak{F}_a(x_1,\ldots,x_n)$ by

$$\mathfrak{F}_a = \sum_{\substack{b \geq a \\ \text{flat}(b) \text{ refines } \text{flat}(a)}} x_1^{b_1} \cdots x_n^{b_n}, \tag{2.2}$$

where $b \geq a$ means $b_1 + \cdots + b_k \geq a_1 + \cdots + a_k$ for all $k = 1,\ldots,n$.

To facilitate virtual objects as defined below, we extend notation and set

$$\mathfrak{F}_{\varnothing} = 0. \tag{2.3}$$

The *run decomposition* of a reduced expression $\rho$, denoted by $(\rho^{(k)}|\cdots|\rho^{(1)})$, partitions $\rho$ into increasing sequences of maximal length. For example, the run decomposition of $(5,6,3,4,5,7,3,1,4,2,3,6)$, a reduced expression for 41758236, is $(56|3457|3|14|236)$.

**Definition 2.3.** For a reduced expression $\rho$ with run decomposition $(\rho^{(k)}|\cdots|\rho^{(1)})$, set $r_k = \rho_1^{(k)}$ and, for $i < k$, set $r_i = \min(\rho_1^{(i)}, r_{i+1} - 1)$. Define the *weak descent composition of* $\rho$, denoted by $\text{des}(\rho)$, by $\text{des}(\rho)_{r_i} = |\rho^{(i)}|$ and all other parts are zero if all $r_i > 0$ and $\text{des}(\rho) = \varnothing$ otherwise.

We say that $\rho$ is *virtual* if $\text{des}(\rho) = \varnothing$. For example, $(5,6,3,4,5,7,3,1,4,2,3,6)$ is virtual since $r_1 = 0$, but the weak descent composition for $(6,7,4,5,6,8,4,2,5,3,4,7)$, a reduced expression for 152869347, is $(3,2,1,4,0,2)$. Note the reversal from the run decomposition to the descent composition.

**Theorem 2.4.** *For $w$ any permutation, we have*

$$\mathfrak{S}_w = \sum_{\rho \in R(w)} \mathfrak{F}_{\text{des}(P)}, \tag{2.4}$$

*where the sum may be taken over non-virtual reduced expressions $\rho$.*

For example, from Figure 2, we have seven non-virtual elements, giving

$$\mathfrak{S}_{153264} = \mathfrak{F}_{(0,3,1,0,1)} + \mathfrak{F}_{(2,2,0,0,1)} + \mathfrak{F}_{(1,3,0,0,1)} + \mathfrak{F}_{(0,3,2,0,0)} + \mathfrak{F}_{(2,2,1,0,0)} + \mathfrak{F}_{(1,3,1,0,0)} + \mathfrak{F}_{(2,3,0,0,0,)}.$$

# 3 Reduced diagrams

A *diagram* is a finite collection of cells in $\mathbb{Z} \times \mathbb{Z}^+$. The *weight* of a diagram $D$, denoted by $\text{wt}(D)$, is the weak composition whose $i$th part is the number of cells in row $i$ of $D$ if all cells have positive row index and $\varnothing$ otherwise. A diagram $D$ is *virtual* if $\text{wt}(D) = \varnothing$.

Given a reduced expression $\rho$, we construct a labeled diagram $\mathbb{D}(\rho)$ such that the row reading word of $\mathbb{D}(\rho)$ is $\rho$ and $\text{wt}(\mathbb{D}(\rho)) = \text{des}(\rho)$.

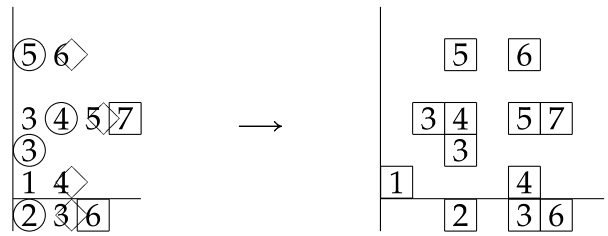

**Figure 3:** The diagram for the reduced expression $(5,6,3,4,5,7,3,1,4,2,3,6)$.

**Definition 3.1.** The *diagram of a reduced expression* $\rho$, denoted by $\mathbb{D}(\rho)$, is constructed as follows. Place values of $\rho^{(i)}$ consecutively from smallest to largest in row $r_i$, as defined in Definition 2.3. Group cells together: begin with highest (then smallest, if tied) ungrouped entry, say $i$, search the next row down for $i-1$ in which case you take it and continue, otherwise search for $i$ in which case you end the group, otherwise continue to the next row down. Maintaining the order within rows, push cells to the right until all entries in each group lie in the same column.

For example, Figure 3 shows the diagram for $(5,6,3,4,5,7,3,1,4,2,3,6)$. Two diagrams are equivalent if they differ by swapping columns that preserves row order.

**Definition 3.2.** A *reduced diagram* is a positive integer filling of a diagram such that

(i) entries are at least as large as the row index;
(ii) rows strictly increase from left to right;
(iii) columns form increasing intervals from bottom to top;
(iv) any $i$ that lies strictly left of an $i+1$ is weakly above it;
(v) reading entries $i$ from left to right strictly descends;
(vi) the row reading word is reduced.

Condition (vi) has several equivalent formulations in terms of restrictions on cells.

**Definition 3.3.** A reduced diagram is *quasi-Yamanouchi* if the leftmost cell of a row has entry equal to its row index or has a cell immediately above and weakly right of it.

Denote the set of quasi-Yamanouchi reduced diagrams of shape $w$ by $\mathrm{QRD}(w)$.

For example, Figure 4 shows $\mathrm{QRD}(153264)$. Note that exactly seven elements are non-virtual, and they have weights that agree with the descent compositions for Figure 2. Moreover, the row reading words of the diagrams in Figure 4 are given by the words in Figure 2, respectively. It turns out that the diagram itself uniquely determines the values that can make it a reduced diagram. In particular, we have the following.

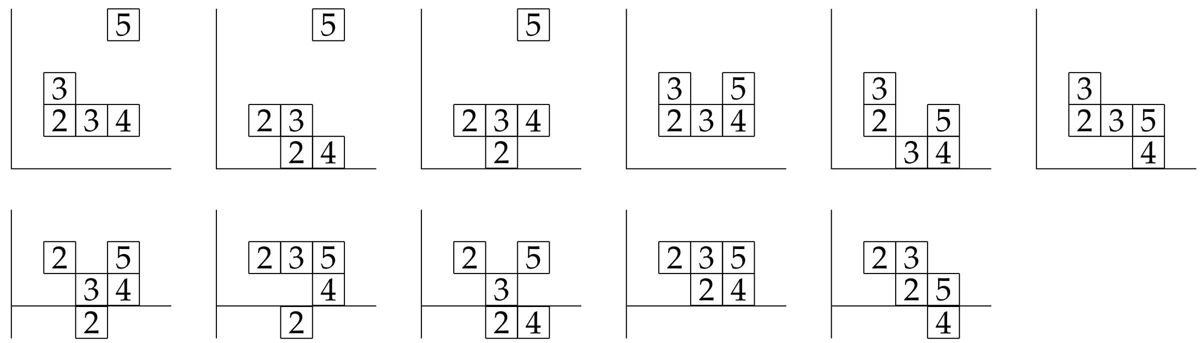

**Figure 4:** The quasi-Yamanouchi reduced diagrams for 153264.

**Theorem 3.4.** *The row reading word is a bijection* $\mathrm{QRD}(w) \xrightarrow{\sim} R(w)$ *that takes weights to weak descent compositions. In particular, the Schubert polynomial* $\mathfrak{S}_w$ *is given by*

$$\mathfrak{S}_w = \sum_{D \in \mathrm{QRD}(w)} \mathfrak{F}_{\mathrm{wt}(D)}, \tag{3.1}$$

*where the sum may be taken over non-virtual quasi-Yamanouchi reduced diagrams for* $w$.

# 4 Balanced tableaux

We transform the reduced diagram model for Schubert polynomials into bijective fillings of the Rothe diagram of a permutation by pushing cells up to the Rothe diagram shape.

**Definition 4.1.** The *Rothe diagram* of a permutation $w$, denoted by $\mathbb{D}(w)$, is given by

$$\mathbb{D}(w) = \{(i, w_j) \mid i < j \text{ and } w_i > w_j\} \subset \mathbb{Z}^+ \times \mathbb{Z}^+. \tag{4.1}$$

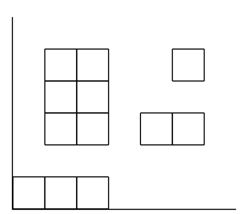

**Figure 5:** The Rothe diagram for 41758236.

Edelman and Greene [3] introduced balanced labelings of the diagrams in order to enumerate reduced expressions. Fomin, Greene, Reiner, and Shimozono [4] generalized this, though their proofs rely on complicated results from geometry. We recover those results with a simple bijection.

**Definition 4.2.** A *standard balanced tableau* is a bijective filling of a Rothe diagram with entries from $\{1, 2, \ldots, n\}$ such that for every entry of the diagram, the number of entries to its right that are greater is equal to the number of entries above it that are smaller.

Denote the set of standard balanced tableaux on $\mathbb{D}(w)$ by $\mathrm{SBT}(w)$.

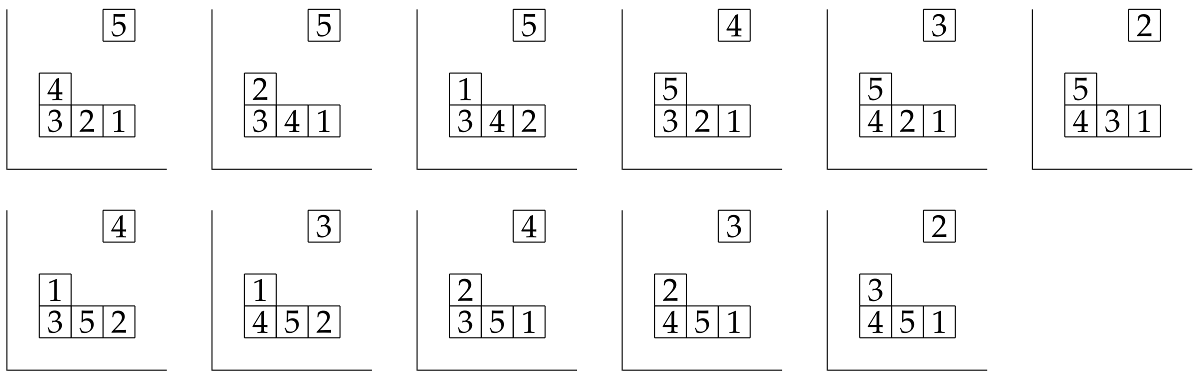

**Figure 6:** The standard balanced tableaux for 153264.

**Definition 4.3.** The *descended diagram* of a standard balanced tableau $R$, denoted by $\mathbb{D}(R)$, is the diagram obtained as follows: find the lowest (smallest if tied) $i$ that lies above some $j > i$; push the cell containing $i$, as well as smaller entries below $i$, down; if, in doing this, $i$ jumps below some $k > i$ in the same column, then swap everything in the column of $k$ with the column of the first entry to its right that is larger; continue until the reverse row reading word is the identity.

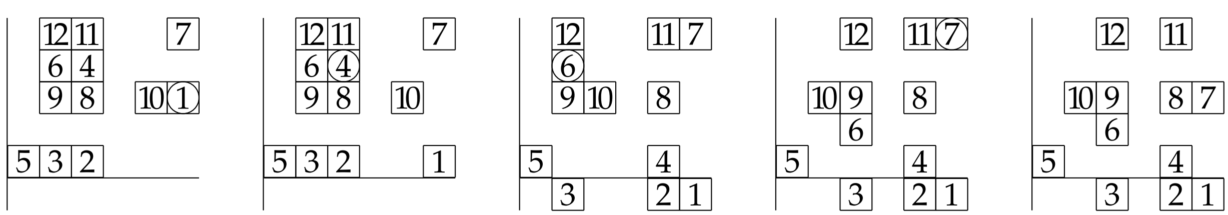

**Figure 7:** Constructing the descended diagram of a balanced tableau for 41758236.

For example, see Figure 7. The shape that results from the descended diagram of a balanced tableau for $w$ is the shape of a necessarily unique reduced diagram for $w$. For example, the descended diagrams for the standard balanced tableaux in Figure 6 are shown in Figure 8. Compare this with Figure 4.

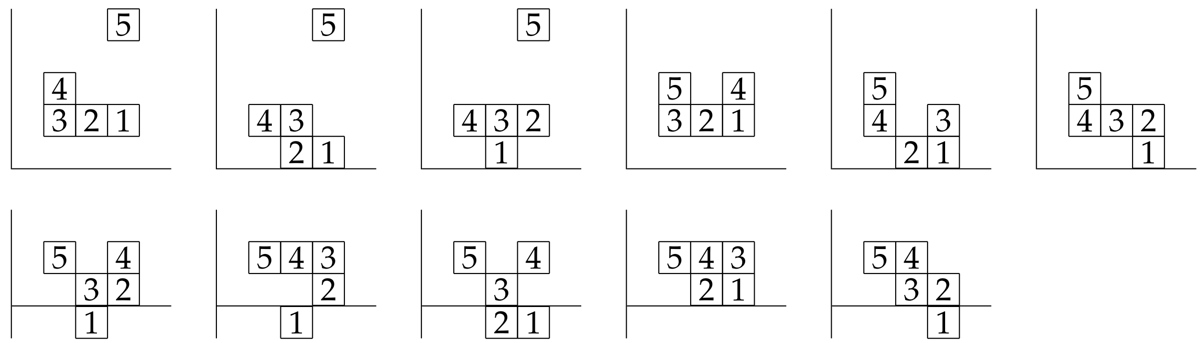

**Figure 8:** Descended diagrams for the standard balanced tableaux of shape 153264.

**Theorem 4.4.** *Descended diagrams give a bijection between standard balanced tableaux and quasi-Yamanouchi reduced diagrams. In particular, the Schubert polynomial for $w$ is given by*

$$\mathfrak{S}_w = \sum_{R \in \mathrm{SBT}(w)} \mathfrak{F}_{\mathrm{des}(R)}, \tag{4.2}$$

*where the sum may be taken over non-virtual standard balanced tableaux of shape $w$.*

# 5 Rothe tableaux

The braid relation on reduced expressions results in the columns moving right on reduced diagrams and in the balanced condition on tableaux. To make diagrams that better resemble the Rothe diagram, we make the following definition for tableaux.

Say that $i$ is *inverted with a row below* if that row has an entry $k$ in the column of $i$ and an entry $j$ immediately right of $k$ such that $i < k, j$. Say that $j$ is *blocked by a column to the left* if there exist $k, l$ in the same row below $j$, with $k$ in the left column and $l$ immediately right of $k$ such that $j \leq k, l$, and, letting $h$ be the entry in the row of $k$ and column of $j$ (or take $h = 0$ if none exists), $j \geq h$.

**Definition 5.1.** A *standard Rothe tableau* is a bijective filling of a Rothe diagram with entries from $\{1, 2, \ldots, n\}$ such that

(i) given $i$ above $k$ in the same column, either $i > k$ or $i$ is inverted with the row of $k$;

(ii) given $i$ left of $j$ in row $r_0$, say in columns $c_i < c_j$, either $i \geq j$ or there exist rows $r_1 > \cdots > r_m$ where the entry $j_k$ in row $r_k$ and column $c_j$ (or take $j_k = 0$ if no entry exists) satisfies $j = j_0 > j_1 > \cdots > j_{m-1} > i \geq j_m$ and each $j_k$ is blocked by row $r_{k-1}$.

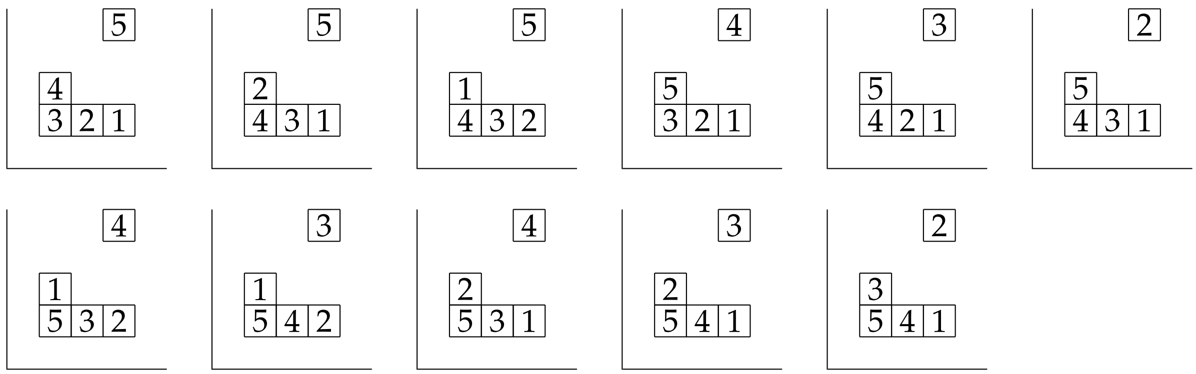

**Figure 9:** The standard Rothe tableaux for 153264.

We denote the set of standard Rothe tableaux of shape $w$ by $\mathrm{SRT}(w)$.

The *run decomposition of a standard Rothe tableau* $T$ is $\tau = (\tau^{(k)}|\ldots|\tau^{(1)})$, where $\tau$ is the word $n\cdots 21$ broken between $i+1$ and $i$ when $i+1$ lies weakly right of $i$ in $T$. For example, the run decomposition for the standard Rothe tableau in Figure 10 is $(12\ 11|10\ 987|6|54|321)$.

**Definition 5.2.** For a standard tableau $T$, let $(\tau^{(k)}|\ldots|\tau^{(1)})$ be the run decomposition of $T$. Set $t_k = \mathrm{row}(\tau_1^{(k)})$ and, for $i < k$, set $t_i = \min(\mathrm{row}(\tau_1^{(i)}), t_{i+1} - 1)$. Define the *weak descent composition of* $T$, denoted by $\mathrm{des}(T)$, by $\mathrm{des}(T)_{t_i} = |\tau^{(i)}|$ and all other parts are zero if $t_i > 0$ for all $i$, and set $\mathrm{des}(T) = \varnothing$ otherwise.

For example, the standard Rothe tableau in Figure 10 is virtual because $r_1 = 0$, but moving cells up and right one position will give a standard Rothe tableau for 152869347 which will have weak descent composition $(3,2,1,4,0,2)$.

**Definition 5.3.** The *descended diagram* of a standard Rothe tableau $T$, denoted by $\mathbb{D}(T)$, is the diagram obtained as follows: find the smallest $i$ that lies above some $j > i$; push $i$ down to the nearest available position, jumping over larger entries and pushing down entries less than $i$; continue until the reverse row reading word is the identity.

Notice that for a standard Rothe tableau $T$, we have $\mathrm{des}(T) = \mathrm{wt}(\mathbb{D}(T))$. Moreover, if, when descending diagrams of the standard balanced tableaux in Figure 8, we do not swap columns when an entry jumps a larger entry, we precisely obtain the diagrams in Figure 11.

**Definition 5.4.** The *unbalancing* of a standard balanced tableau $R$, denoted by $\mathbb{U}(R)$, is the tableau obtained as follows. Let $D = \mathbb{D}(R)$, and let $\mathrm{sort}(R)$ denote $R$ with its rows sorted into decreasing order. Beginning with the lowest row and the leftmost cell therein of $D$, move the cell left until it lies in the same column as in $\mathrm{sort}(R)$. If the cell cannot

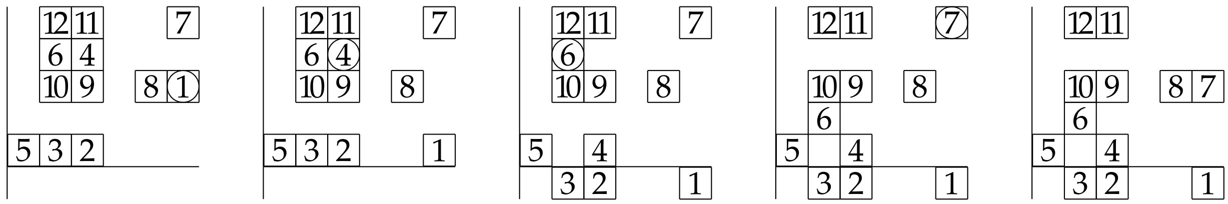

**Figure 10:** Constructing the descended diagram of a Rothe tableau for 41758236.

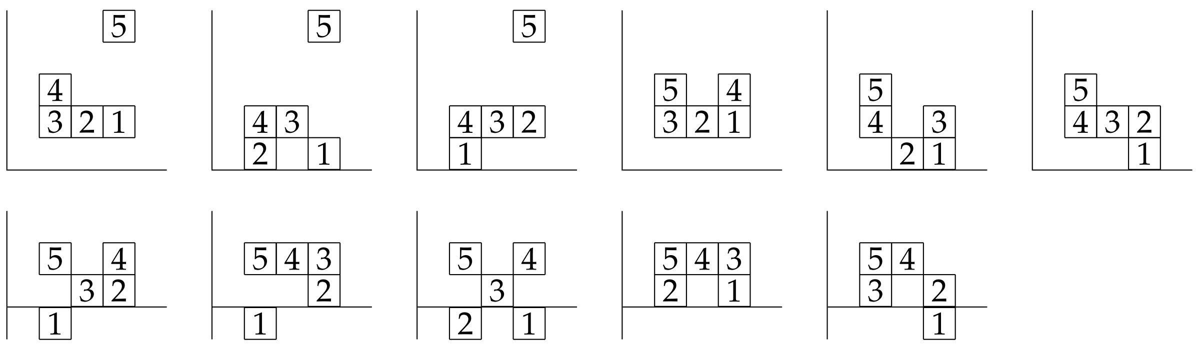

**Figure 11:** The descended diagrams for the standard Rothe tableaux of shape 153264.

move left, then scan down the column for the highest cell below it that can move left and move that cell instead. Continue right to the end of the row, then continue with the next row above. Once finished, ascend the cells back to the Rothe diagram shape.

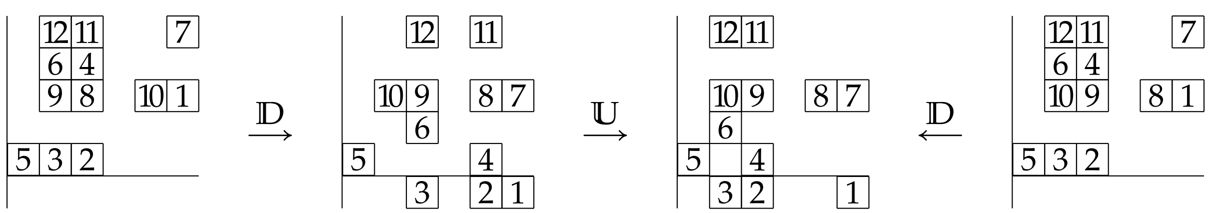


**Figure 12:** Unbalancing a balanced tableau to a Rothe tableau for 41758236.

For an example of unbalancing, see Figure 12.

**Theorem 5.5.** *Unbalancing gives a* des*-preserving bijection between standard balanced tableaux and standard Rothe tableaux. In particular, the Schubert polynomial for w is given by*

$$\mathfrak{S}_w = \sum_{T \in \mathrm{SRT}(w)} \mathfrak{F}_{\mathrm{des}(T)}, \tag{5.1}$$

*where the sum is over non-virtual standard Rothe tableaux of shape w.*

# 6 Kohnert diagrams

Kohnert [5] conjectured the following rule for computing a Schubert polynomial from the Rothe diagram. Select a nonempty row and push the rightmost cell of that row down to the first open position below it. Say that any diagram obtained in such a way is a *Kohnert diagram*, and denote the set of Kohnert diagrams for $w$ by $\mathrm{KD}(w)$.

**Theorem 6.1** (Kohnert's rule [5]). *The Schubert polynomial for $w$ is given by*

$$\mathfrak{S}_w = \sum_{D \in \mathrm{KD}(w)} x^{\mathrm{wt}(D)}, \tag{6.1}$$

*where the sum may be taken over non-virtual Kohnert diagrams for $w$.*

Kohnert and many others attempted a proof of this rule, and Winkel ultimately published two proofs [8, 9], though neither of these proofs is widely accepted by the community given the intricate inductive argument that obfuscates the main idea of the proof. Thus it remains open to give a simple, direct proof of Kohnert's rule.

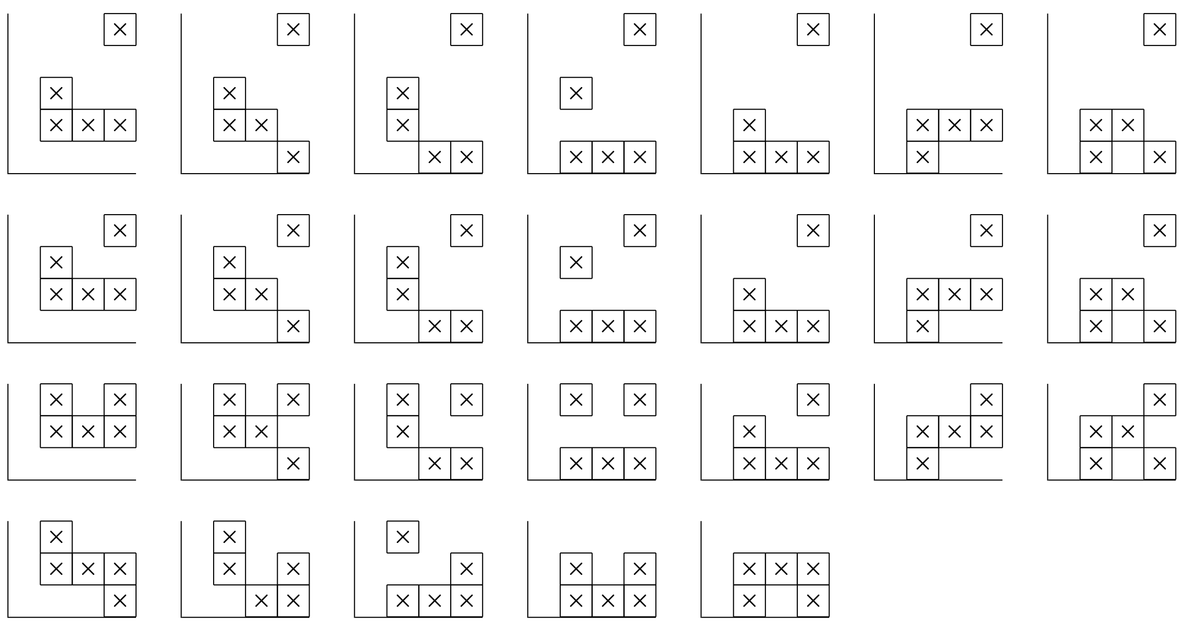

**Figure 13:** The Kohnert diagrams of shape 153264.

It is easy to see that the descended diagram of a standard Rothe tableau has the shape of a Kohnert diagram by following the pushing procedure.

**Definition 6.2.** A *semi-standard Rothe tableau* is a positive integer filling of a Rothe diagram satisfying the conditions of Definition 5.1 such that no entry exceeds its row index.

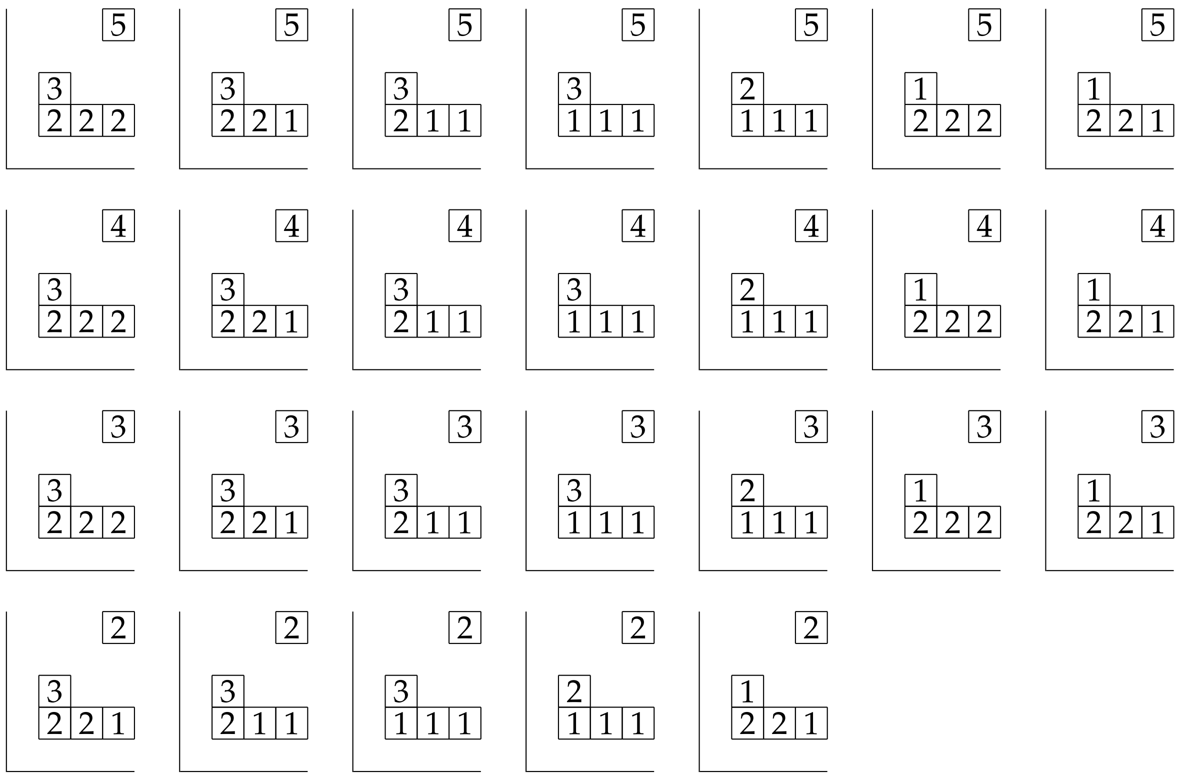

**Figure 14:** The semi-standard Rothe tableaux of shape 153264.

For example, see Figure 14. The obvious standardization map that turns a semi-standard filling into a standard filling and takes the weight to the weak descent composition results in the tableaux in Figure 9, counted with multiplicity.

**Lemma 6.3.** *For $w$ any permutation, we have*

$$\sum_{T \in \mathrm{SRT}(w)} \mathfrak{F}_{\mathrm{des}(T)} = \sum_{T \in \mathrm{SSRT}(w)} x^{\mathrm{wt}(T)}, \tag{6.2}$$

*where the left sum may be taken over non-virtual standard Rothe tableaux for $w$.*

We can extend Definition 5.3 to semi-standard Rothe tableaux as follows.

**Definition 6.4.** The *descended diagram* of a semi-standard Rothe tableau $T$, denoted by $\mathbb{D}(T)$, is the diagram obtained by pushing all entries $i$ down to row $i$.

By selecting the smallest $i$ not in its row and the rightmost if tied, the Rothe tableau conditions ensure that the descended diagram is a Kohnert diagram. Moreover, this process is reversible for any Kohnert diagram.

**Theorem 6.5.** *Descended diagrams give a bijection between semi-standard Rothe tableaux and Kohnert diagrams. In particular, the Schubert polynomial for $w$ is given by*

$$\mathfrak{S}_w = \sum_{D \in \mathrm{KD}(w)} x^{\mathrm{wt}(D)}. \tag{6.3}$$

*That is, Kohnert's rule for Schubert polynomials, holds.*

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